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QUATERNIONS.

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(Continued from page 13.)

2. HAMILTON sought to establish a system which would, at the outset, give a clear interpretation to the square roots of negatives without introducing considerations so expressly geometrical as those which involve the idea of an angle, and was thus lead to consider Algebra as the SCIENCE OF PURE TIME; an essay upon which he read before the Royal Irish Academy in the year 1843.* From this as a starting point he proceeded by a logical process to the establishment of his new system.† But Hamilton, like many other inventors, was not fortunate in the popular presentation of his system. His lectures upon the subject—delivered in Trinity College, Dublin, in 1848 and subsequent years—were, in 1853, published in book form of 736 pages;‡ but the style is so peculiar, being diffuse and hesitating as if he *labored* to make his readers understand, and the arrangement being such as to separate parts of essential principles, that, probably, comparatively few of its readers, without other aids, have mastered its principles. Still the work is remarkably thorough, and remains a monument to the genius of the man. As these articles are designed for those who have little or no knowledge of the subject, we shall seek the most direct and simple manner of explaining its principles. Frequent references will be made to Hamilton's Lectures, for the convenience of those who have access to that work and who may desire to compare results and processes.

3. The system may be developed from the two following lemmas, here stated in the form of postulates:—

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*Proceedings of the Royal Irish Academy, Vol. XVII, Part II, pp. 293-422.

†Hamilton's Lectures, Preface, p. 2. ‡Now out of print.

1.—Let it be granted, that the relations between the sides of a plane triangle, represented *in length and direction* respectively by the letters α, β, γ, may be expressed by the equation

$$\alpha = \beta + \gamma. \tag{1}$$

2.—Let it be granted that, if three mutually perpendicular lines be represented respectively by ai, bj, k,—where i, j, k, are each unity in length—the operation of turning ai about k as an axis through a quadrant to coincide in direction with j and comparing the length of bj with ai, may be represented by the equation

$$\frac{b}{a}k = \frac{bj}{ai}. \tag{2}$$

Developing these equations according to the laws involved in the respective propositions, and introducing from time to time suitable notation for expressing certain operations, we may build up the entire system of Quaternions. And, in order to make it practical, it will only be neceseary to interpret the results in accordance with the fundamental principles und the signification of the notation employed. This method was recognized as a correct logical process by Hamilton;* but as it is very arbitrary, we prefer to show in what sense the lemmas may be true.

4. Let the reader imagine that he travels three miles due south, thence northwesterly five miles reaching a point four miles due west of the starting point. Let S represent one mile in a due southerly direction ($N.\ W$), one mile in the northwesterly direction, and W one mile due west. Then the operation of traveling may be expressed thus,

$$3S + 5(N.W),$$

the result of which—in regard to the initial and terminal positions—is the same as if one traveled due west four miles, or $4W$. Hence, in regard to these positions, we may write

$$4W = 3S + 5(N.W).$$

This equation is not algebraic, though its form appears to be such; for the letters W, &c., represent both length and direction. The entire distance traveled in one case is eight miles, and in the other four miles; which distances are unequal. Still, there is equality in the sense that each member represents the *result* of two independent operations. The addition in the second member is not algebraic, since, in algebra, two quantities having incongruous units cannot be added, but addition here implies a succeeding similar operation—one followed by another step.

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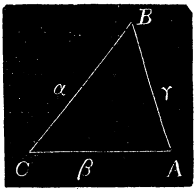
*Lectures, p. 38.

5. In a similar manner, if the length and direction of AB be represented by γ, which direction may be parallel to some fixed line in space, and need have no reference to the points of the compass; and similarly β represent CA and α, CB; then may the position of B in reference to C be represented in the same manner by the equation



$$\alpha = \beta + \gamma.^*$$

An extension of this principle gives, for the relation between two adjacent angles of a polygon, the equation

$$\alpha = \beta + \gamma + \delta + \&c.$$

The signs = and + as here used have a broader meaning than in algebra, but they include their full signification as used in that science.

6. A VECTOR literally implies a transference of a point a given distance in a given direction.† Vectors which have the same direction are parallel.‡ Parallel vectors whose lengths are equal are said to be equal; and if their lengths are unequal they are multiples of each other. Thus one of the parallel sides of a trapezoid is a multiple vector of the other. If the length of a vector be unity, it is called *a unit vector*. A vector being positive in one direction, will be negative in the opposite direction.** The positive sign of a vector may be fixed arbitrarily, but being fixed it must retain that sign in that direction throughout any particular discussion. Co-initial vectors are such as radiate from one point.

If, in the triangle ABC, the transference be from B completely round the triangle and positive right-handed, we have

$$\gamma + \beta + \alpha = 0.^{\dagger\dagger}$$

If positve left-handed, we have

$$\alpha + \beta + \gamma = 0,$$

or, passing around right-handed,

$$-\gamma - \beta - \alpha = 0;$$

which is the same as the preceding with the signs changed.

7. THE COMMUTATIVE principle consists in interchanging the terms. Thus in the triangle ABC—vectors positive left-handed—we have, beginning at A, $\quad \gamma + \alpha + \beta = 0,$

and beginning at C, $\quad \beta + \gamma + \alpha = 0.$

Hence the terms may be interchanged as in algebra.

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*Lectures, pp. 29, 30, 44. †Ibid. p. 15. **Ibid. p. 51. ††Ibid. p. 144.

‡Some authors of elementary geometry define parallel lines as those which have the same direction, and though it is strictly correct, and possibly the best definition ever given, yet, experience with beginners shows that it is not the most elementary.

8. THE ASSOCIATIVE principle consists in combining the terms in different groups, thus

$$(\alpha + \beta) + \gamma = \alpha + (\beta + \gamma) = (\alpha + \gamma) + \beta,$$

which is the same as in algebra.*

9. A TENSOR is a numerical factor by which a vector is multiplied.† Thus, in the expressions $b\alpha$, $-c\beta$, 3γ, the factors b, $-c$, and 3, are tensors. A tensor is sometimes denoted by the letter T, thus, in the preceding expressions $T\alpha = b$, $T\beta = -c$, $T\gamma = 3$. If α be a unit vector, and $\alpha_1 = b\alpha$ be the entire vector then we have

$$\alpha_1 = T\alpha\,(\alpha).$$

10. The law of signs in relation to the transposition of terms is the same as in algebra. For in the triangle ABC, if we have, as given in eq'n (1), the position of B in reference to C,

$$\alpha = \beta + \gamma,$$

then for A in reference to C, we have

$$\beta = \alpha - \gamma,$$

or, of A in reference to B,

$$-\gamma = -\alpha + \beta;\ddagger$$

which results may be obtained from (1) by the transposition of terms.

11. Co-planar vectors are such as are in one plane. Any three co-planar vectors which give the relation

$$a\alpha + b\beta + c\gamma = 0,$$

will form a closed triangle. For, constructing a plane triangle the lengths of whose sides are the lengths respectively of $a\alpha$, $b\beta$, $c\gamma$ (or if α, β, γ, be unit-vectors, then the lengths will be a, b, c, respectively) the triangle may be placed in such a position that its sides will be parallel to the respective vectors.

12. If as the result of analysis we have

$$\Sigma\alpha + \Sigma\beta = 0,$$

then must $\qquad \Sigma\alpha = 0$, and $\Sigma\beta = 0$.

For $\Sigma\alpha$ is typical of a set of parallel vectors, and in a developed form may be represented by $m\alpha + n\alpha + l\alpha +$&c.; and similarly $\Sigma\beta = p\beta + q\beta +$ &c., another set of parallel vectors; and since two vectors cannot enclose a space, or, more generally, since two vectors not parallel have no common unit, each must be zero in order that their sum shall be zero.

This may be illustrated by a familiar example. Thus, if one travels north and south, different distances at different times; also east and west in

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*Lectures, pp. 102, 103. † Ibid. pp. 57, 87, 88, 114. ‡ Ibid. pp. 30, 101, 102.

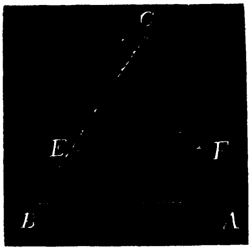
a similar manner, he will not end his journey *at the starting point* unless the sum of the north and south distances be zero, and the sum of the east and west distances, also zero; and if these separately are zero his terminal position will coincide with the initial, and the resultant of all the distances will be zero.

13. The principles now established are sufficient for the solution of a large class of problems. Operations under equation (1) are called the Addition and Subtraction of vectors.

Applications.

1—If the corresponding sides of two triangles are proportional, the triangles are similar.

Let ABC and FEC be two triangles in which we have



$$\frac{AB}{FE} = \frac{BC}{EC} = \frac{AC}{FC},$$

then will the corresponding angles be equal and hence the triangles will be similar. One side CE of one triangle may be made to coincide in direction with the side CB of the other. Let the lengths of the sides be $CE = l$, $EF = m$, $FC = n$, $CB = x$, $BA = y$, $AC = z$; and the unit-vectors of the corresponding sides α, β, γ, α, δ, ε; observing that the unit-vector for CB is the same as for CE.

The triangle CEF gives

$$l\alpha + m\beta + n\gamma = 0,$$

and CBA,

$$x\alpha + y\delta + z\varepsilon = 0.$$

Eliminating α gives

$$xm\beta - yl\delta + xn\gamma - zl\varepsilon = 0.$$

But the sides being proportional, we have

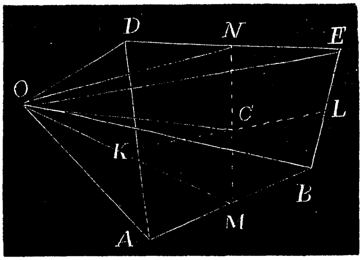
$$\frac{l}{x} = \frac{m}{y} = \frac{n}{z}, \text{ or } xm = yl, \text{ and } xn = zl;$$

which reduces the former equation to

$$y(\beta - \delta) + z(\gamma - \varepsilon) = 0.$$

The relation of y to z is fixed by the condition of the problem, and β and γ are known; hence the equation can be satisfied only by making $\delta = \beta$ and $\varepsilon = \gamma$; hence BA must be parallel to EF and CA to CF, and hence $\angle CEF = \angle CBA$, and $\angle CFE = \angle CAB$. In the figure CA coincides with CF, but, according to the analysis, it is only necessary that the sides be parallel.

2.—To find the centre of gravity of four equal particles situated anywhere in space.



Let A, B, E, D, represent the position of the particles. The centre of gravity of those at A and B will be at M, the middle point of the line AB. Similarly, the centre of gravity of D and E will be at N, the middle point of DE. Joining M and N, the centre of gravity of all the particles will be at C, the middle point of MN.

To find the position of this point, take any point O as the origin of vectors, and draw the vectors $OA = \alpha$, $OB = \beta$, $OE = \gamma$, $OD = \delta$, and $OC = \mu$. Then as vectors $OA + AB = OB$,

or $\quad \alpha + AB = \beta;$

$\therefore\ AB = \beta - \alpha,$

and $\quad AM = \frac{1}{2}(\beta - \alpha).$

Similarly, $\quad DN = \frac{1}{2}(\gamma - \delta).$

Also $\quad ON = OD + DN,$

or, substituting for DN, $\quad = \delta + \frac{1}{2}(\gamma - \delta),$

reducing, $\quad = \frac{1}{2}(\gamma + \delta).$

Similarly, $\quad OM = \frac{1}{2}(\alpha + \beta).$

Also, $\quad ON + NM = OM,$

or $\quad NM = OM - ON;$

substituting, $\quad = \frac{1}{2}(\alpha + \beta) - \frac{1}{2}(\gamma + \delta)$

$= \frac{1}{2}(\alpha + \beta - \gamma - \delta);$

hence $\quad NC = \frac{1}{4}(\alpha + \beta - \gamma - \delta).$

Finally, $\quad OC = ON + NC,$

or $\quad \mu = \frac{1}{2}(\gamma + \delta) + \frac{1}{4}(\alpha + \beta - \gamma - \delta)$

$= \frac{1}{4}(\alpha + \beta + \gamma + \delta);$

that is, the vector to the centre of gravity is the mean of all the vectors. This point is also called the mean point of the polygon formed by joining the several particles.* It will be observed that the mode of solution consists in reaching a point by two independent routes. The particles, are not confined to one plane, and the quadrilateral $ABEF$ formed by joining the particles, may be a warped surface. An interpretation of the last result shows that—If a polygon be formed, beginning at the origin, whose successive sides are equal and parallel to the respective vectors and the closing side be divided into the same number of equal parts as there are particles, the centre of gravity will be at the first point of division from the origin.

(To be continued.)

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*Hamilton's Lectures, p. 458.